Stable solutions using the Euler approximation

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A minor modification of the standard Euler approximation for the solution of oscillatory problems in mechanics yields solutions that are stable for arbitrarily large number of iterations, regardless of the size of the iteration interval. The period of a nonlinear oscillator converges rapidly to its exact value as the size of the iteration interval is decreased. In two dimensions, closed orbits are given for the two-body Kepler problem and the restricted three-body problem can be iterated indefinitely to produce space-filling orbits. In this new approximation, the difference ΔE between the initial energy and the energy after n iterations is bounded, oscillatory, and zero when averaged over half a cycle of the motion.

I. INTRODUCTION

The rapidly increasing availability of microcomputers and programmable calculators has stimulated interest in simple numerical methods that students can use to solve physics problems.[1,2] Of particular interest is the numerical solution of mechanics problems, because these problems vividly demonstrate the dynamic nature of Newton's second law.

For a particle of unit mass moving in one dimension, the position x at time t is related to the force $F(x)$ by the first-order differential equations

$$\dot{v} = F(x), \quad \dot{x} = v. \tag{1}$$

This set of equations is solved numerically by converting it to a set of difference equations and then iterating. The Euler approximation, the midpoint approximation, and the half-step approximation are three linear approximations commonly used to solve Eqs. (1), although the half-step approximation is much better than the other two.

In this paper we present a new linear approximation that is better than the half-step approximation. Furthermore, we present an analytic proof of the superiority of the new approximation. This proof shows that the usual explanation[1] of why the half-step approximation is better than the Euler approximation is wrong. The difference between these approximations is not how well they approximate the derivative at each point, but on how well they approximate the first integral of the motion.

Let D be the time interval between successive iterations, so that the time t_n of the nth iteration is

$$t_n = t_0 + nD. \tag{2}$$

Let x_n and v_n be the values of x and v on the nth iteration, and let $F_n = F(x_n)$. Then the Euler, or *first-point*, approximation (FPA) is

$$\left.\begin{aligned} v_{n+1} &= v_n + F_n D \\ x_{n+1} &= x_n + v_n D \end{aligned}\right\} \text{FPA.} \tag{3}$$

We call this the first-point approximation because, in the equation for x_{n+1}, the velocity v_n at the beginning of the iteration interval is used to estimate the average velocity during the interval.

The *midpoint approximation* (MPA) is

$$\left.\begin{aligned} v_{n+1} &= v_n + F_n D \\ x_{n+1} &= x_n + (1/2)(v_n + v_{n+1})D \end{aligned}\right\} \text{MPA.} \tag{4}$$

This approximation seems more reasonable than the FPA, and in fact it does give exact results when F is a constant. However, for more general problems, it is not substantially better than the FPA. Both approximations are equally poor, because their errors increase with each iteration. In the case of a one-dimensional oscillator, for instance, these approximations give solutions whose maxima increase with each cycle.

The *half-step approximation* (HSA) is

$$\left.\begin{aligned} v_{1/2} &= v_0 + (1/2)F_0 D \\ v_{n+1/2} &= v_{n-1/2} + F_n D \\ x_{n+1} &= x_n + v_{n+1/2} D \end{aligned}\right\} \text{HSA.} \tag{5}$$

As can be seen, the velocity is iterated in between the position. Although it is considerably more difficult to explain than the FPA, it is no more difficult to program. It is, however, a very much better approximation than the FPA or MPA, because its error is bounded. This means that Eqs. (5) can be iterated indefinitely, without the error ever growing beyond a fixed value. The usual explanation of this approximation's superiority is that, by using the velocity in the middle of the interval, it estimates the average velocity better than the FPA. This explanation is not correct, because the MPA, which does a similar thing, is no better than the FPA. Moreover, a still better approximation is obtained by using the velocity at the end of the interval.

The *last point approximation* is

$$\left.\begin{aligned} v_{n+1} &= v_n + F_n D \\ x_{n+1} &= x_n + v_{n+1} D \end{aligned}\right\} \text{LPA.} \tag{6}$$

As unpromising as it looks, this approximation is superior to the HSA. It yields solutions that, for oscillatory problems, have errors that are bounded, oscillatory, and zero when averaged over half a period. The LPA yields stable solutions that can be iterated indefinitely. As a result, an undergraduate—and even a high school student—can obtain very accurate solutions to advanced mechanics problems.

The LPA was discovered quite by accident by Abby Aspel,[3] a student at Newton North High School (Newton, MA). She was working on a computer program for the Kepler problem, and had written the correct program for

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Table I. Period for oscillations of unit amplitude in the force $F = -x - bx^3$. The exact values are from Chen,[5] and the other values are found from the last point approximation [Eqs. (8)] run for N cycles with an iteration interval D.

b	1	4	9
exact	4.768	3.181	2.294
$D = 0.2, N = 10$	4.72	3.10	2.14
$D = 0.05, N = 4$	4.76	3.175	2.265
$D = 0.001, N = 0.5$	4.768	3.179	2.294

the FPA. The equivalent program in one-dimension is

```
100 F = (function of X)
110 X = X + V*D
120 V = V + F*D
130 PRINT X, V
140 GOTO 100
```

FPA. (7)

Although this program worked fairly well for half an orbit (about 14 iterations), the solution started to diverge during the second half. Thinking she had made a mistake, Aspel interchanged statements 110 and 120. The new program is

```
100 F = (function of X)
110 V = V + F*D
120 X = X + V*D
130 PRINT X, V
140 GOTO 100
```

LPA. (8)

Unbelievably, this program produced a closed orbit in 28 iterations. Comparing Eqs. (3) and (6) to Eqs. (7) and (8) we can see that interchanging the order in which x and v are computed changes the velocity used in the position equation from v_n to v_{n+1}.

Since the difference between the FPA and the LPA is only the interchange of two lines in a computer program, many people must have used the LPA without realizing it. Perhaps because the LPA gives such good results, they never suspected that they had made a mistake (or a discovery). The standard books on numerical analysis[4] do not mention the LPA, and as far as I can determine, the proof given in Sec. III of the validity of the approximation is entirely new. This proof explains why these four different linear approximations yield solutions of greatly different accuracy.

In Sec. II we present some test calculations that demonstrate the accuracy of the last point approximation and in Sec. III we prove analytically that the error oscillates about an average value of zero.

II. CALCULATIONS

As a first test of the LPA, we shall find the period T of a particle moving under the influence of the nonlinear force

$$F = -kx - bx^3. \tag{9}$$

In terms of the potential energy

$$U = (1/2)kx^2 + (1/4)bx^4$$

the period is given by[5]

$$T = 2\sqrt{2}\int_0^A [E - U(x)]^{-1/2}dx$$
$$= (4/\sqrt{k\lambda})K([(\lambda - 1)/2\lambda]^{1/2}),$$

where A is the amplitude, the parameter λ is

$$\lambda = bA^2/k + 1,$$

and K is the complete elliptic integral of the first kind. Chen gives the exact value of T for several values of λ. Here we shall find the period in a half-hour exercise that a high school student can perform.

The program in Eqs. (8) was run using the function in Eq. (9) with $k = 1$ and different values of b. In all cases the initial conditions were $x = 1, v = 0$, so $A = 1$. Table I shows the results obtained using different values of the iteration interval D.

The program in Eqs. (8) is *perfectly stable.* It can be run for hundreds of cycles with no observable error accumulation. To find T, the program is run for N cycles, and the elapsed time divided by N.

Table I shows that with $D = 0.2$ and $N = 10$, the LPA gives periods that differ from the exact values by between 1 and 7%. With $D = 0.05$ and $N = 4$, the errors vary from 0.01 to 1%, and with $D = 0.001$ and $N = 0.5$, the results are exact to four significant figures. Thus the accuracy of the LPA is equal to its precision. The LPA calculates the period to within the iteration interval D, because it has no accumulation of error, even after thousands of iterations.

To better understand this approximation, it is helpful to plot the motion of the nonlinear oscillator in phase space. With an Apple II-plus microcomputer this can be done directly on a TV screen. Figure 1 shows such a plot of v against x for the case $b = 4$ and $D = 0.05$. Figure 1 is the way the TV screen looks after ten cycles have been plotted. To within the resolution of the screen (about one percent), all the computed cycles fall on the same curve.

The amazing stability of the LPA indicates that something exact is going on. In the FPA, the phase-space curve spirals outward, because the energy of the system increases monotonically. Clearly there is no such monotonic increase

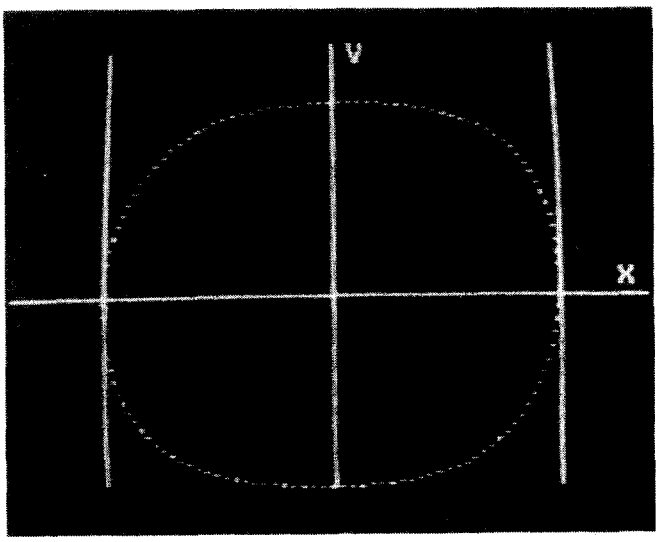


Fig. 1. Phase space diagram of the motion of a particle of unit mass in the force $F = -x - 4x^3$. The curve was calculated using the LPA with initial conditions $x = 1$ and $v = 0$, and with the iteration interval $D = 0.05$.

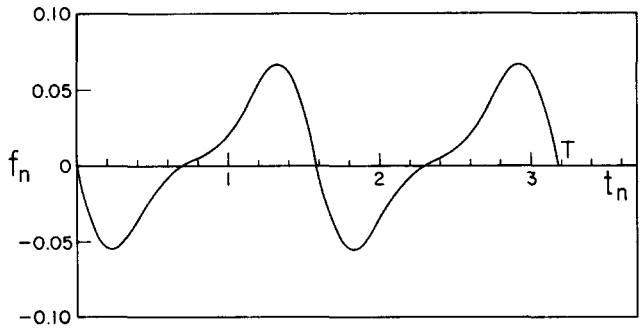


Fig. 2. The relative energy $f_n = (E_n - E_0)/E_0$ plotted against $t_n = nD$ for the oscillator in Fig. 1.

in the LPA. Figure 2 is a plot of the relative energy change

$$f_n = \frac{\Delta E}{E} = \frac{E_n - E_0}{E_0}$$

against $t_n = nD$ for one cycle of the motion shown in Fig. 1. Here

$$E_n = (1/2)v_n^2 + U(x_n).$$

Figure 2 shows that in the LPA, the energy oscillates about E_0 with a period one-half the period of the oscillator. In Sec. III we shall prove that this is a general property of the approximation. The error in calculating the time for the oscillator to move from $x = A$ to $x = 0$ is almost exactly the negative of the error in calculating the time from 0 to $-A$. Thus in half a cycle, the errors of the approximation cancel to a large extent.

The LPA was discovered while working on the numerical solution of the Kepler problem. Because the approximation self-corrects in half a cycle, it produces closed orbits with only 100 iterations per cycle ($D = T/100$). Figure 3 is a typical elliptical orbit as it appears on the TV screen after dozens of cycles. In this case, $D = 0.2$, $x_0 = -4$, $y_0 = 0$, $v_{x0} = 0$, $v_{y0} = 1.8$, and the potential energy is $-GM/r$, with $GM = 10$. The initial energy is -0.88, and the maximum value of $|f_n|$ is 0.025. The calculated period is $T = 27$ and the calculated semimajor axis is $a = 5.75$. The exact values for these quantities are

$$a = -GM/(2E) = 5.68,$$

and

$$T = 2\pi a^{3/2}(GM)^{-1/2} = 26.9.$$

Again we see that the calculation is exact to within the precision set by the choice of the iteration interval.

The stability of the LPA for orbit calculations means that it can be used to study the effects of atmospheric drag on earth-orbiting satellites and the effect of small departures from an inverse-square force on the orbits of planets.

Finally, in Fig. 4, we show some orbits of a planet moving in the gravitational field of two fixed stars of equal mass. The details of this calculation, including a computer program that uses an FPA-like approximation, have been given by Wild.[2] To get the results shown here, it is necessary only to interchange statements 00-15 with statements 16-66 in his program.[6].

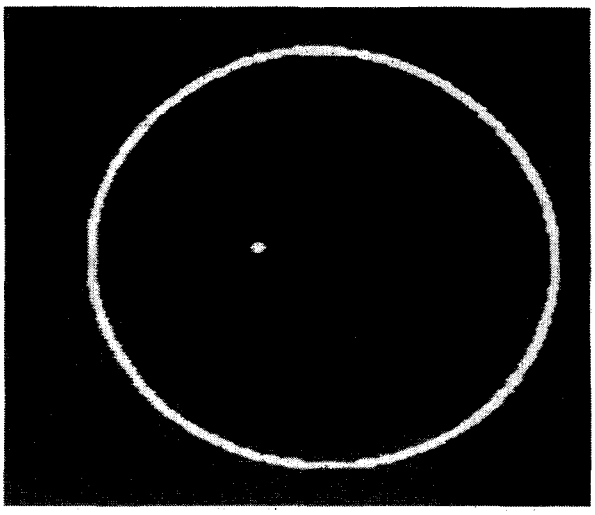
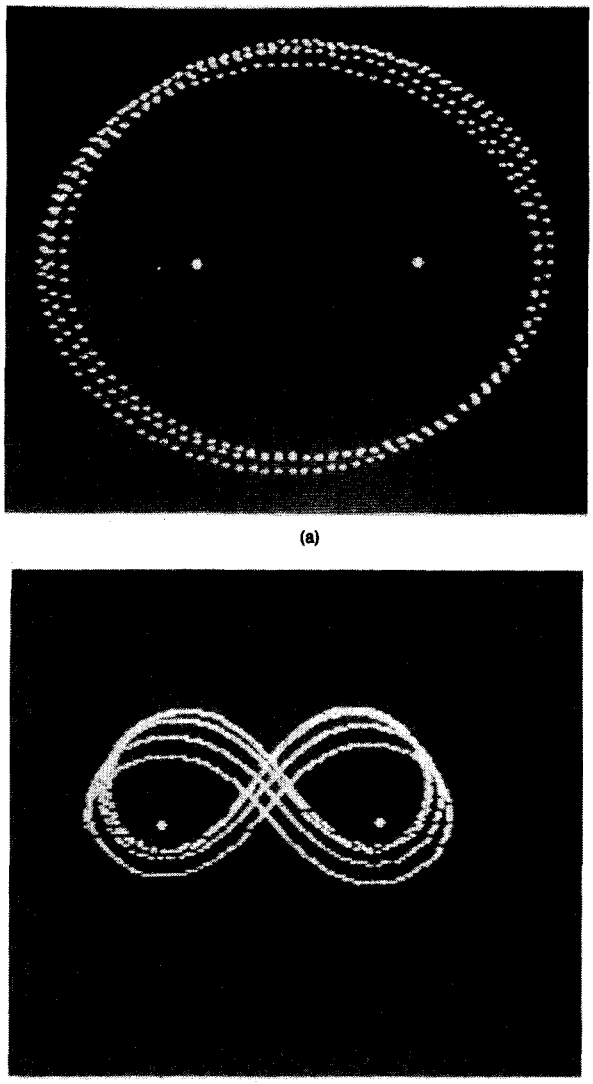


Fig. 3. Orbit of a planet moving about a star with mass $GM = 10$. The initial radius is 4 and the initial (tangential) speed is 1.8. The orbit was calculated using the LPA with initial condition $x_0 = -4$, $y_0 = 0$, $v_{x0} = 0$, $v_{y0} = 1.8$, and with the iteration interval $D = 0.2$.



(a)

(b)

Fig. 4. Stable space-filling orbits for a planet moving about two fixed stars with masses $GM = 10$. The parameters for these orbits are given in Table II. (a) Open loop. (b) Figure eight.

Table II. Parameters for two stable three-body orbits. This table gives the initial conditions, iteration interval, and calculated period for the two orbits shown in Fig. 4. The two stars each have a mass $GM = 10$ and are separated by a distance of 5.93. The left-hand star is at the origin.

orbit type	x_0	y_0	v_{x0}	v_{y0}	D	T
open loop	−4	0	0	2	0.2	20.4
figure eight	−2	0	0	2.3	0.05	10.7

As pointed out by Wild, the FPA cannot be used to follow the history of an orbit for more than two cycles. Even during this limited time interval, the energy of the system is constantly changing, so the orbits calculated by the FPA do not represent the true motion to any meaningful degree. In contrast, with the LPA, the energy of the system oscillates within narrow limits, except when the planet comes too close to a star. During a close encounter, the energy changes by 50% or more. In a moderately close encounter, the motion remains bounded, and the energy returns to its initial value as the planet moves away. In an extremely close encounter, the energy becomes positive, and the planet shoots off to infinity. To avoid these unphysical results, Fig. 4 shows two orbits that never make a close encounter of either kind. The parameters of these orbits are given in Table II.

Figure 4(a) shows the open-loop orbit after several revolutions. Because of the large radius of this orbit, an iteration time of 0.2 is sufficient to keep the energy constant to within 2.5%. Thus the orbit shown in Fig. 4(a) is calculated as accurately, and as easily, as the elliptical orbit in Fig. 3. This orbit has been calculated for over 100 revolutions (12000 iterations) without incident. In time, the orbit completely fills an oval ring around the two stars.

Figure 4(b) shows the figure-eight orbit after several revolutions. Because of the smaller radius, an iteration interval of 0.05 is needed to keep the energy constant to within 10%. This orbit has been calculated for over 60 revolutions (12000 iterations) without incident. In time, it too fills a twisted strip that encircles the two stars.

Thus unlike the FPA, the LPA can find the stable three-body orbits as well as interesting irregularly looped orbits. By making the second star a large planet moving in a fixed orbit about the first star, the LPA should be able to make reliable perturbation calculations.

III. PROOF OF STABILITY

In Sec. II we showed empirically that the LPA conserves energy, on the average, for oscillatory motion. In this section we shall prove this analytically, thus putting the LPA on a sound theoretical basis.

Without loss of generality, we let $v_0 = 0$. Then, iterating Eqs. (6) n times, we get

$$v_n = (F_0 + F_1 + \cdots + F_{n-1})D = S_{n-1}, \tag{10}$$

$$x_{n+1} = x_n + S_n D, \tag{11}$$

where

$$S_n = D \sum_{j=0}^{n} F_j. \tag{12}$$

Had we used the FPA [Eqs. (3)], the last term in Eq. (11) would be $S_{n-1}D$ instead of S_nD. This is where the critical difference in the two approximations comes in.

With $v_0 = 0$, the change in kinetic energy between $t_0 = 0$ and $t_n = nD$ is

$$\Delta K_n = K_n - K_0 = (1/2)S_{n-1}^2. \tag{13}$$

The change in potential energy over this same time interval is

$$\Delta U_n = -\int_{x_0}^{x_n} F(x)dx.$$

Using the trapezoid rule to evaluate this integral, we get

$$\Delta U_n = -(1/2)\sum_{i=0}^{n-1}(F_i + F_{i+1})(x_{i+1} - x_i)$$

$$= -(1/2)D\sum_{i=0}^{n-1}(F_i + F_{i+1})S_i$$

$$= -(1/2)D^2\sum_{i=0}^{n-1}\sum_{j=0}^{i}(F_i + F_{i+1})F_j.$$

The key point here is that, since j runs from 0 to i (instead of $i-1$), ΔU_n has the same squared terms as ΔK_n. Separating out these terms, we can rewrite ΔU_n as

$$\Delta U_n = -(1/2)D^2\left(\sum_{i=0}^{n-1}F_i^2 + \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}F_iF_j + \sum_{i=1}^{n}\sum_{j=0}^{i-1}F_iF_j\right)$$

$$= -(1/2)D^2\left(\sum_{i=0}^{n-1}F_i^2 + 2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}F_iF_j + F_n\sum_{j=0}^{n-1}F_j\right)$$

$$= -(1/2)S_{n-1}^2 - (1/2)DF_nS_{n-1}. \tag{14}$$

Thus the change in total energy as given by the LPA is

$$\Delta E_n = \Delta K_n + \Delta U_n = -(1/2)DF_nS_{n-1}$$

$$= -(1/2)DF_nv_n, \tag{15}$$

where we again use Eq. (10).

For oscillatory motion, v_n is zero at the turning points and F_n is zero at the equilibrium point. Thus ΔE_n goes through zero four times in each cycle, which proves that it oscillates with a period equal to half that of the oscillator, as demonstrated in Fig. 2. Furthermore, the maximum value of ΔE_n is bounded, since both F_n and v_n are bounded. Finally, the average of ΔE_n over half a cycle is

$$\langle \Delta E_n \rangle = \frac{D^2}{T}\sum_{n=0}^{(1/2)T/D}F_nv_n \simeq \frac{D}{T}\int_0^{(1/2)T}Fv\,dt$$

$$= -(D/T)\{U[(1/2)T] - U(0)\} = 0,$$

since U has the same value at each turning point. This proves that, on the average, the LPA conserves energy for oscillatory motion.

Applying the same method of analysis to the FPA, we find that the expression for ΔE_n has the term

$$\sum_{i=0}^{n-1}F_i^2$$

in it. Since this term increases monotonically with n, the error in the energy increases with n. On the other hand, in the HSA, the energy change is found to be

$$E_n = -(1/8)D^2(F_0^2 - F_n^2).$$

With the initial condition $v_0 = 0$ that we are using, $F_0^2 \geq F_n^2$. Thus ΔE_n oscillates between 0 and $-(1/8)D^2F_0^2$ twice during each cycle. The energy in the HSA is bounded as in

the LPA, but the average energy change during a cycle is not zero. This means that the HSA will not be as accurate as the LPA in calculating the period of an oscillator.

This analysis shows that the difference between the different linear approximations is not how well they approximate the derivative at each point, but how well they approximate the first integral of the motion.

IV. CONCLUSIONS

The last point approximation is a simple but powerful method for solving problems of oscillatory motion. The proof of the approximation's stability is simple enough to give to upperclass undergraduates, and the approximation itself is simple enough to be used by a high school student.

With such an elegant tool at our disposal, the three-body problem may become as much a part of the introductory physics curriculum as the inclined plane.[7]

[1]R. M. Eisberg, *Applied Mathematical Physics with Programmable Pocket Calculators* (McGraw-Hill, New York, 1976).

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[7]A. Cromer, *Three-Body Orbits* (EduTech, Newton, MA, 1981). Software package for solving the three-body problem on a Apple II-plus computer.